Vere-Jones' self-similar branching model

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Motivated by its potential application to earthquake statistics as well as for its intrinsic interest in the theory of branching processes, we study the exactly self-similar branching process introduced recently by Vere-Jones. This model extends the ETAS class of conditional self-excited branching point-processes of triggered seismicity by removing the problematic need for a minimum (as well as maximum) earthquake size. To make the theory convergent without the need for the usual ultraviolet and infrared cutoffs, the distribution of magnitudes m' of daughters of first-generation of a mother of magnitude m has two branches $m'<m$ with exponent $\beta-d$ and $m'>m$ with exponent $\beta+d$, where β and d are two positive parameters. We investigate the condition and nature of the subcritical, critical, and supercritical regime in this and in an extended version interpolating smoothly between several models. We predict that the distribution of magnitudes of events triggered by a mother of magnitude m over all generations has also two branches $m'<m$ with exponent $\beta-h$ and $m'>m$ with exponent $\beta+h$, with $h=d\sqrt{1-s}$, where s is the fraction of triggered events. This corresponds to a renormalization of the exponent d into h by the hierarchy of successive generations of triggered events. For a significant part of the parameter space, the distribution of magnitudes over a full catalog summed over an average steady flow of spontaneous sources (immigrants) reproduces the distribution of the spontaneous sources with a single branch and is blind to the exponents β, d of the distribution of triggered events. Since the distribution of earthquake magnitudes is usually obtained with catalogs including many sequences, we conclude that the two branches of the distribution of aftershocks are not directly observable and the model is compatible with real seismic catalogs. In summary, the exactly self-similar Vere-Jones model provides an attractive new approach to model triggered seismicity, which alleviates delicate questions on the role of magnitude cutoffs in other non-self-similar models. The new prediction concerning two branches in the distribution of magnitudes of aftershocks could be tested with recently introduced stochastic reconstruction methods, tailored to disentangle the different triggered sequences.

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I. INTRODUCTION

Stochastic branching processes describe well a multitude of phenomena [1,2] from chain reactions in nuclear and particle physics, material rupture, fragmentation and earthquake processes, to population and biological dynamics, epidemics, economic, and social cascades and so on. Branching processes are also of particular interest because deep connections have been established with critical phenomena. In branching processes, various quantities exhibit power law distributions at criticality. This includes the distributions of cluster sizes, of the number of generations before extinction and of durations.

Branching processes become critical when the average death rate is exactly compensated by the average growth rate. At criticality, branching processes become asymptotically self-similar, which translates for instance into an asymptotic power law tail for the distribution of cluster sizes. Such scale invariance is a general characteristic of systems at the critical point of a phase transition or of a bifurcation. But self-similarity in general holds only (i) at criticality and (ii) asymptotically, i.e., at scales much larger than the microscopic mesh or elementary branch scale.

Vere-Jones has recently introduced a class of self-similar branching processes which is exactly self-similar for a broad range of parameters (of nonzero measure), that is, far from criticality and for all scales [3]. Vere-Jones self-similar branching process is derived from a class of exactly self-similar random measures which generalize the class of stable purely atomic completely random measures. The underlying idea is that a change of scale is balanced by a change in mass, making the measure, and the branching process, self-similar at all scales and off-criticality. This is possible only for branching processes with continuous masses or "marks." Then, the model has a natural application to describe earthquake triggering, which is the example we adopt in the following to formulate the problem and present our results, without loss of generality.

The concrete example proposed by Vere-Jones is the "self-similar ETAS" (epidemic-type aftershock sequence) model, which is a self-similar extension of the initial non-self-similar standard ETAS model [6,21]. The main statistical

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properties of the standard ETAS model are reviewed in Refs. [4,5]. The ETAS model describes the rate $\lambda(\vec{r},t,m)$ of events (earthquakes for instance) at position $\vec{r}$ at time t and of mass or mark (magnitude) m resulting from spontaneous sources ("immigrants") and from all past events. The model is defined in terms of the conditional Poisson intensity $\lambda(\vec{r},t,m)$ which is a function of all past events. The standard ETAS model describes earthquake triggering by past ones within the framework of branching theory and takes into account the interplay between the exponential productivity law and the Gutenberg-Richter (GR) law of event sizes. A series of papers has shown that the standard ETAS model gives a reasonable description of the statistics of seismic clustering [6], of foreshocks [7,8] and aftershocks [4,5] triggered by other earthquakes, of the empirical Båth's law for the largest aftershock of a given sequence [9,10] and of the statistics of seismic rates [11]. The standard ETAS model thus constitutes a powerful null hypothesis to test against other models [6].

The standard ETAS model is perhaps the simplest of a much larger class of models embodying the physics of triggered seismicity. Its simplicity results from its assumption of complete decoupling between the Gutenberg-Richter distribution of event sizes, productivity law, time and space interaction kernels. Already in 1988, Ogata proposed an extension allowing for magnitude-dependent time and space kernels and taking into account spatial anisotropic effects [6]. Using statistical likelihood methods, this extension has recently been shown to be superior to the standard ETAS model to account for empirical earthquake clustering [12]. Ogata *et al.* have also extended ETAS to characterize regional features of seismic activity in and around Japan, by allowing the parameter values to vary from place to place [13]. In the same vein, we have shown that the empirical distribution of seismic rate in California can be described adequately by ETAS model only when taking into account a strongly nonuniform fractal distribution of spontaneous sources [11]. Ouillon and Sornette have considered a multifractal model of triggered seismicity which predicts magnitude-dependent exponents for the Omori law of aftershock decay rates, in good agreement with empirical data [14,15]. Unfortunately, few of these extensions have allowed yet for a full theoretical understanding of the stationary properties of the resulting catalogs of triggered events.

All these models assume the existence of a minimum magnitude m_0 below which earthquakes do not trigger other earthquakes. This assumption is necessary to regularize the theory which would otherwise become divergent as small earthquakes dominate collectively in the generation of progenies [19,20]. Sornette and Werner have noted that the magnitude m_d of completeness of a seismic catalog has no reason to be the same as the magnitude m_0 of the smallest triggering earthquake, since m_d results from instrumental limitations while m_0 should be associated with the physics of earthquake rupture [22]. The existence of the ultraviolet cutoff m_0 has actually observable consequences in constraining the calibration of the models [22]. In addition, the difference between m_0 and m_d leads to a reformulation of the models into renormalized branching processes with apparent branching ratio and apparent rate of immigrants [23]. This implies in particular that previous estimates of the clustering parameters of real catalogs which did not address the difference between m_0 and m_d may significantly underestimate the true values, for instance, an observed fraction of 55% of aftershocks in catalogs is renormalized into a true value of 75% by properly taking into account the difference between m_0 and m_d.

But does m_0 really exist? Introducing an ultraviolet cutoff to regularize a theory is never welcome, neither aesthetically nor for its physical implications. This is such a central theme in theoretical physics that the dependence of physical quantities on the chosen cutoffs (especially the ultraviolet cutoffs) is the main focus of the theory of renormalization group. This motivates us to study the new exactly self-similar Vere-Jones model, which provides the most natural modification of the class of self-excited branching models allowing us to remove the cutoff. The question then becomes the following: by modifying the ETAS model, is the new exactly self-similar model compatible with empirical observations?

We thus present a detailed theoretical understanding of some statistical properties of the self-similar Vere-Jones model, which extends the standard ETAS model by removing the need of cutoffs. In an effort to clearly formulate some universal key features of the self-similar Vere-Jones model, we shall consider mostly the statistics of total numbers of aftershocks triggered by spontaneous sources and their distribution. In our present study, we integrate over space and time to focus on global properties, such as the total number of events triggered by a given event. In looking at conditions for stability, if one integrates out various dimensions of the branching model (time, space), one must be careful before assuming that results for the simplified model carry over to the full model. New phenomena may appear in higher dimensions and may not be reflected in the behavior of lower-dimensional counterparts. Sufficient conditions for subcriticality in the cruder model are probably still sufficient for the higher model, but more delicate conditions might break down or need to be refined (Vere-Jones, private communication).

The organization of the paper is as follows. In the next section, we introduce a model more general than the Vere-Jones "self-similar ETAS" model, which we call the "generalized Vere-Jones ETAS model." This then allows us to obtain the standard ETAS model and the self-similar ETAS model as two special cases, which puts in perspective the structure of Vere-Jones' self-similar ETAS model. In other words, the generalized version allows us to interpolate smoothly between the standard ETAS model and Vere-Jones' model. We stress that only Vere-Jones' self-similar ETAS model is exactly self-similar, as the generalized Vere-Jones ETAS model requires in general a truncation in the distribution of event magnitudes at some low magnitude threshold m_0 (which is pushed to $-\infty$ for Vere-Jones' self-similar ETAS model). Section III presents the general theoretical treatment in terms of generating probability functions (GPF) which predicts in particular the exact form of the magnitude distributions. Section IV explores the phase diagram of all three models by identifying the conditions for subcriticality, criticality, and supercriticality. Section V derives the distributions of earthquake magnitudes for Vere-Jone's self-similar model. Section VI presents a summary of our predictions, a discussion of their consequences and their possible empirical tests.

II. VERE-JONES' SELF-SIMILAR ETAS MODEL

A. Definition of the "generalized Vere-Jones ETAS model"

Let us consider an event of magnitude m which may trigger another event of magnitude m' according to a conditional Poisson process with intensity $\lambda(m,m')$, so that the total average rate of production of events of magnitude m' is

$$\lambda(m') = \sum_m \lambda(m,m') \tag{1}$$

where

$$\lambda(m,m') = \kappa e^{am-\beta m'-d|m-m'|}, \tag{2}$$

gives the average number of events of magnitude m' triggered directly (on first generation) by an event of magnitude m. The space and time dependence of the mother (or triggering event) of magnitude m and daughter (or triggered event) of magnitude m' have disappeared from the expression of $\lambda(m,m')$ due to the integration over space and time, so that we are concerned only here with total numbers in a fixed space-time window. The exponential in (2) contains three contributions,

(1) e^{am} describes the exponentially growing productivity of a source as a function of its magnitude m [19,20] (in other words, e^{am} is proportional to the average number of first-generation progenies of the source m).

(2) $e^{-\beta m'}$ is the so-called Gutenberg-Richter distribution of the magnitudes of the first-generation triggered events. We sometimes use the term "aftershocks" to refer to these events.

(3) The new term $e^{-d|m-m'|}$, compared with previous models of the ETAS class, describes a binding or localization of the magnitude m' of triggered events in the neighborhood of the ancestor's magnitude m. In other words, this term means that daughters' magnitudes keep a memory of the size of their mothers, large (small) daughters come more probably from large (small) mothers for $d>0$. This could be associated with the fact that triggered events occur on patches of the mother's fault rupture with large residual stresses (for the $e^{-d(m-m')}$ branch with $m'<m$) and on faults branching from the mother's rupture (for the branch $e^{d(m-m')}$ with $m'>m$).

The branching model is such that any event can trigger other events according to the rate $\lambda(m,m')$ given by (2). Thus, a given event may give daughters of first generation, which can themselves trigger other events and so on, giving rise to a triggering cascade. The model also considers the existence of spontaneous sources (immigrants), seeding the branching cascades.

B. The ETAS model: $d=0$

For $d=0$, i.e., when the size of daughters is independent (has no memory) of the size of the mother, model (2) recovers the standard ETAS model [4,6,21],

$$\lambda(m,m') = \kappa' e^{a(m-m_0)} \beta e^{-\beta(m'-m_0)}, \tag{3}$$

where the first factor on the right-hand side (rhs) is the so-called exponential productivity law and the second one is the Gutenberg-Richter law of first generation events magnitudes which is normalized ($\int_{m_0}^{\infty} dm' \beta e^{-\beta(m'-m_0)}=1$). A minimum event size m_0 is necessary to make the ETAS model well-defined [22,23]. Indeed, it is not possible to make the model convergent for $m_0 \to -\infty$ (a $-\infty$ magnitude corresponds to a vanishing energy, since the magnitude is proportional to the logarithm of the energy), and an ultraviolet cutoff is necessary [22,23].

The constant factor κ' is derived from the coefficient κ of the generalized Vere-Jones ETAS model (2) as

$$\kappa' = \frac{\kappa}{\beta} e^{(a-\beta)m_0}. \tag{4}$$

The ETAS model is critical when its average branching ratio (equal to the number of progenies averaged over all mothers' magnitudes)

$$n \equiv \int_{m_0}^{\infty} dm \kappa' e^{a(m-m_0)} \beta e^{-\beta(m-m_0)} = \frac{\kappa' \beta}{\beta - a} \tag{5}$$

is unity. The case $n<1$ (respectively $n>1$) corresponds to the subcritical (respectively, supercritical) regime. The condition $a<\beta$ is needed to make the model convergent, as both the borderline $a=\beta$ and the regime $a>\beta$ leads to finite-time singularities with stochastic times [24]. Alternatively, convergence and stationary is obtained by adding an upper magnitude cutoff $m_{\max}$, associated with the empirical bending down of the Gutenberg-Richter law for magnitudes larger than about 8, suggesting that $m_{\max} \approx 8$–9 [25–27]. We shall not consider the influence of this upper cutoff whose impact is rarely felt only at time scales and for earthquake numbers so large to ensure that the largest earthquakes are sampled.

The ETAS model, unlike Vere-Jones model, lacks self-similarity due to both the nongeneric condition $n=1$ for criticality and the existence of a minimum event magnitude m_0 which introduces a characteristic magnitude scale.

C. The self-similar Vere-Jones model: $d>0$ and $a=\beta$

Vere-Jones self-similar ETAS model corresponds to $a=\beta$ and $d>0$ in (2),

$$\lambda(m,m') = \lambda(m-m'), \quad \lambda(m) = \kappa e^{\beta m - d|m|}. \tag{6}$$

The condition $a=\beta$ expresses a balance between the exponential small probability of finding a large mother and its exponentially large productivity. When $a=\beta$ (and in absence of the effect of d), each event magnitude range $[m,m+dm]$ contributes equally to other event triggering, and, in particular, small events are as important to event triggering as are larger ones, a property which seems to be approximately true for earthquakes [19,20]. In the standard ETAS model, the case $a=\beta$ is not possible without the introduction of both an ultraviolet cutoff m_0 and an additional infrared cutoff $m_{\max}$ truncating the Gutenberg-Richter distribution [25–27], which are necessary in order to obtain nondiverging sequences. In Vere-Jones self-similar ETAS model, the condition $a=\beta$ is made possible without cutoffs by the introduction of the parameter $d>0$. Physically, expression (6) (together with the standard Gutenberg-Richter law) can be interpreted by the existence of two branches for the Gutenberg-Richter distribution of the $\sim e^{\beta m}$ events triggered by a given mother of magnitude m,

(1) daughters with magnitude $m' < m$ have their magnitudes m' distributed according to $\sim e^{-(\beta-d)m'}$ while

(2) daughters with magnitude $m' > m$ have their magnitudes m' distributed according to $\sim e^{-(\beta+d)m'}$.

The former (respectively, latter) distribution branch of daughter magnitudes ensures that there are less small (respectively, large) daughters than with $d=0$. This is the origin, as we will make clear below quantitatively, of the possibility to avoid any cutoff m_0 or $m_{\max}$ and still obtain a convergent model. This is a first reason why Vere-Jones' model (6) is self-similar. As we shall see below, the other reason is that there is a finite range of parameters for which Vere-Jones is effectively critical.

III. GENERAL THEORETICAL FORMULATION

A. From Bernoulli to Poisson statistics

One of the main ingredients of both the standard ETAS and Vere-Jones' models of aftershocks triggering is the Poissonian statistics governing the number of first generation aftershocks triggered by some spontaneous source of given magnitude m. In order to obtain a deeper insight into the origin of the underlying Poissonian law, we start with the more general Bernoulli approach to the description of aftershocks triggering statistics.

Consider the Bernoulli version of a generalized model in which each spontaneous source of magnitude m has p independent "possibilities" to trigger some aftershock. The probability that an aftershock of magnitude in the interval $[m', m'+dm']$ is actually triggered along one of these p paths is denoted $D(m,m',p)dm'$. Keeping the approach general, we allow $D(m,m',p)$ to depend on the source magnitude m, the number of channels p and the triggered magnitude m' in an arbitrary way, in all that follows in this section. The next section will then apply our formalism to the specifications (2) and (6), using the standard ETAS version (3) as a reference and point of comparison.

The generating probability function (GPF) of the random number of first generation aftershocks, obeying to the above Bernoulli statistics, is given by

$$\Theta_1(z;m,p) = \left(1 + \int_{m_0}^{\infty} dm' D(m,m',p)(z-1)\right)^p. \quad (7)$$

Here, m_0 is the smallest magnitude of possible triggered aftershocks, which will be pushed to $-\infty$ in the self-similar Vere-Jones version.

An essential assumption of the models studied here is that triggered events of first generation act themselves as sources which trigger their own aftershocks according to the same laws. Let us call $\Theta(z;m',p)$ the GPF of the random number of events of all generations triggered by a first-generation daughter with magnitude m'. Then, the GPF of the number of aftershocks triggered over all generations by a given mainshock of magnitude m is solution of

$$\Theta(z;m,p) = \left(1 + \int_{m_0}^{\infty} dm' D(m,m',p)(z\Theta(z;m',p)-1)\right)^p, \quad (8)$$

obtained from (7) by replacing z by $z\Theta(z;m',p)$ to express that each branch has the same statistical cascade properties.

Consider the limiting intensity of aftershocks triggering

$$\lim_{p\to\infty} pD(m,m',p) = \lambda(m,m'), \quad (9)$$

and its associated Poissonian GPF limit

$$\Theta(z;m) = \lim_{p\to\infty} \Theta(z;m,p). \quad (10)$$

Exrpession (8) leads to the nonlinear integral equation

$$\Theta(z;m) = \exp\left(\int_{m_0}^{\infty} dm' \lambda(m,m')[z\Theta(z;m')-1]\right). \quad (11)$$

It will be useful in the sequel to study the statistics of those triggered events with magnitude larger than some threshold μ, whose corresponding GPF for their numbers is denoted $\Theta(z;m,\mu)$. The equation for $\Theta(z;m,\mu)$ is obtained from (11) by replacing $\Theta(z;m)$ by $\Theta(z;m,\mu)$ on both the left-hand side (lhs) and rhs and by replacing z on the rhs by

$$H(\mu-m') + zH(m'-\mu) = \begin{cases} 1 & \text{if} \quad m' < \mu, \\ z & \text{if} \quad m' > \mu, \end{cases} \quad (12)$$

where $H(x)$ is unit step function, equal to 1 if $x > 0$ and 0 otherwise. Replacing z by (12) just means that only those first generation aftershocks, whose magnitudes are larger than μ, are counted. Their subsequent cascade above the magnitude level μ is accounted for by replacing $\Theta(z;m')$ by $\Theta(z;m',\mu)$. This leads to

$$\Theta(z;m,\mu) = \exp\left(\int_{m_0}^{\infty} dm' \lambda(m,m')(\Theta(z;m',\mu)-1) + (z-1)\int_{\mu}^{\infty} dm' \lambda(m,m')\Theta(z;m',\mu)\right). \quad (13)$$

B. Distribution of events magnitudes

One of the simplest and most informative statistical characteristics of branching processes is the average number $\langle R\rangle(m,\mu)$ of events of magnitude above μ triggered by some spontaneous source of magnitude m,

$$\langle R\rangle(m,\mu) = \left.\frac{d\Theta(z;m,\mu)}{dz}\right|_{z=1}. \quad (14)$$

Using expressing (13) in (14), we find that $\langle R\rangle(m,\mu)$ satisfies to linear integral equation

$$\langle R\rangle(m,\mu) = \int_{m_0}^{\infty} \lambda(m,m')\langle R\rangle(m',\mu)dm' + \langle R_1\rangle(m,\mu), \tag{15}$$

where

$$\langle R_1\rangle(m,\mu) = \int_{\mu}^{\infty} \lambda(m,m')dm' \tag{16}$$

is the average number of first generation aftershocks with magnitudes larger than μ.

We assume that the spontaneous sources constitute a stationary point process with Poisson statistics, with an average number of spontaneous sources during a time interval τ equal to $\omega\tau$. Let us furthermore denote $p(m)$ the probability density function (PDF) of the magnitude of the random sources. Then, the total average number of events (including the spontaneous sources and all their offsprings over all generations) during the time interval τ with magnitudes larger than μ is equal to

$$\omega\tau[\langle R\rangle(\mu) + Q(\mu)], \tag{17}$$

where

$$\langle R\rangle(\mu) = \int_{m_s}^{\infty} \langle R\rangle(m,\mu)p(m)dm \tag{18}$$

is the average number of events of all generations with magnitudes larger than or equal to μ which are triggered by the spontaneous sources of all possible magnitudes above some lower threshold m_s defined as the smallest magnitude of spontaneous events. In what follows, we assume that the spontaneous sources have their magnitudes distributed according to a GR law

$$p(m) = \chi e^{-\chi(m-m_s)}H(m-m_s), \tag{19}$$

with an exponent χ possibly distinct from those of triggered events. The complementary cumulative distribution function (CDF) of spontaneous source magnitudes then reads

$$Q(\mu) = \int_{\mu}^{\infty} p(m)dm = e^{-\chi(\mu-m_s)}, \quad \mu > m_s. \tag{20}$$

It is natural to introduce a magnitude threshold m_d of catalog completeness, i.e., only events with $m > m_d$ are observed. Then, the total fraction of events above magnitude μ among all observable events in the time window τ is given by the following normalization of (17):

$$F(\mu,m_d) = \frac{\langle R\rangle(\mu) + Q(\mu)}{\langle R\rangle(m_d) + Q(m_d)}. \tag{21}$$

$F(\mu,m_d)$ can be interpreted as the complementary CDF of magnitudes of observable events. The corresponding PDF of observable events is then

$$f(\mu,m_d) = \frac{g(\mu) + p(\mu)}{\langle R\rangle(m_d) + Q(m_d)}, \quad g(\mu) = -\frac{d\langle R\rangle(\mu)}{d\mu}. \tag{22}$$

Expression (17) also allows us to obtain the fraction of triggered events

$$n(\mu) = \frac{\langle R\rangle(\mu)}{\langle R\rangle(\mu) + Q(\mu)}, \tag{23}$$

which is nothing but the average branching ratio [28].

In the sequel, we apply relations (22) and (23) to the standard ETAS model (3) and to the self-similar Vere-Jones model (6).

IV. CRITICALITY CONDITION AND PHASE DIAGRAM

A. General maximum eigenvalue condition

Before going further, it is important to derive the conditions under which the branching process is not explosive, i.e., for which the stationary spontaneous sources point process generates a stationary sequence of triggered events. The condition derives in general from an eigenvalue problem (see Ref. [29] and references therein). In the present case, it is known that Eq. (15) gives bounded stationary solutions if the largest eigenvalue ρ of the corresponding homogeneous equation

$$\rho\mathcal{R}(m) = \int_{m_0}^{\infty} \lambda(m,m')\mathcal{R}(m')dm' \tag{24}$$

is smaller than 1 ($\rho < 1$). This corresponds to the subcritical regime. The condition $\rho = 1$ defines the critical regime and $\rho > 1$ gives the explosive supercritical regime. In our analysis of the eigenvalue problem of Eq. (24), we shall restrict to physically meaningful eigenfunctions $\mathcal{R}(m)$ which are monotonically increasing functions growing no faster than the productivity law $\sim e^{am}$,

$$\lim_{m\to\infty} \mathcal{R}(m)e^{-am} < \infty. \tag{25}$$

An important point should be noted in relation with Vere-Jones's model. While in the standard ETAS model, the total number of triggered events above any arbitrary magnitude m_d diverges at criticality, we will see that the average total number $\langle R\rangle(m_d)$ of observable events is finite in the critical regime of Vere-Jones' model, while the average total number $\langle R\rangle(m_d \to -\infty)$ is itself infinite. Thus, while the process is critical when considering all events of any magnitude, it becomes subcritical for events above a finite threshold m_d. The hallmark of such subcritical regime is that the fraction (23) for observable events is smaller than 1 i.e., $n(\mu) < 1$ for all $\mu > m_d$.

B. Criticality condition for the standard ETAS model

For the standard ETAS model defined in Sec. II B, Eq. (24) reduces to

$$\rho\mathcal{R}(m) = \kappa' e^{a(m-m_0)} \int_{m_0}^{\infty} e^{-\beta(m'-m_0)}\mathcal{R}(m')dm'. \tag{26}$$

Let us look for a solution of this equation in the form

$$\mathcal{R}(m) = Ce^{\delta m}. \tag{27}$$

Substituting (27) in (26) shows that $C \neq 0$ if and only if $\delta = a$ while the corresponding eigenvalue is

$$\rho = \kappa \int_{m_0}^{\infty} e^{(a-\beta)m} dm = \frac{\kappa' \beta}{\beta - a}, \quad (28)$$

which is nothing but the average branching ratio n defined in (5). This recovers the known fact [4] that the condition $\rho = n < 1$ corresponds to the subcritical regime associated with the solution

$$\langle R \rangle (m, \mu) = \frac{\kappa'}{1-\rho} e^{a(m-m_0)-\beta(\mu-m_0)}, \quad (29)$$

of the nonhomogeneous equation (15).

Substituting (29) and (19) into (18) and (22), and assuming for simplicity that $m_s = m_0$, we obtain the distribution of the PDF of the magnitudes μ of observable events

$$f(\mu, m_d) = \frac{\langle R \rangle \beta e^{-\beta(\mu-m_0)} + \chi e^{-\chi(\mu-m_0)}}{\langle R \rangle e^{-\beta(m_d-m_0)} + e^{-\chi(m_d-m_0)}} H(\mu - m_d), \quad (30)$$

where

$$\langle R \rangle = \langle R \rangle (m_0) = \frac{\rho'}{1-\rho}, \quad \rho' = \frac{\kappa' \chi}{\chi - a} \quad (31)$$

is the average of the total number of aftershocks triggered by one spontaneous source of arbitrary magnitude. In the particular case where the GR laws for the magnitudes of the spontaneous sources and of the first generation aftershocks are the same, i.e., if $\chi = \beta$, then the PDF of the magnitudes of observable events given by (30) reduces to the pure GR law $f(\mu, m_d) = \beta e^{-\beta(m-m_d)} H(\mu - m_d)$.

Substituting

$$\langle R \rangle (\mu) = \frac{\rho'}{1-\rho} e^{-\beta(\mu-m_0)}, \quad Q(\mu) = e^{-\chi(\mu-m_0)} \quad (32)$$

into expression (23) yields

$$n(\mu) = \frac{\rho'}{\rho' + (1-\rho) e^{(\beta-\chi)(\mu-m_0)}}. \quad (33)$$

If, as before, $\chi = \beta$ ($\rho' = \rho$), then $n(\mu)$ does not depend on μ and is equal to the average branching ratio $n = \rho$.

C. Criticality condition for the self-similar Vere-Jones model

For the self-similar Vere-Jones model defined in Sec. II C, the homogeneous equation (24) reduces to

$$\rho \mathcal{R}(m) = \int_{-\infty}^{\infty} \lambda(m - m') \mathcal{R}(m') dm', \quad (34)$$

where $\lambda(m)$ is given by expression (6). We search again a solution for $\mathcal{R}(m)$ of the form (27), which yields the following eigenvalue $\rho(\delta)$ as a function of δ (shown in Fig. 1):

$$\rho(\delta) = \int_{-\infty}^{\infty} \lambda(m) e^{-\delta m} dm = \frac{2\kappa d}{d^2 - (\delta - \beta)^2} = \frac{s}{1 - (\delta - \beta)^2 / d^2}, \quad (35)$$

where

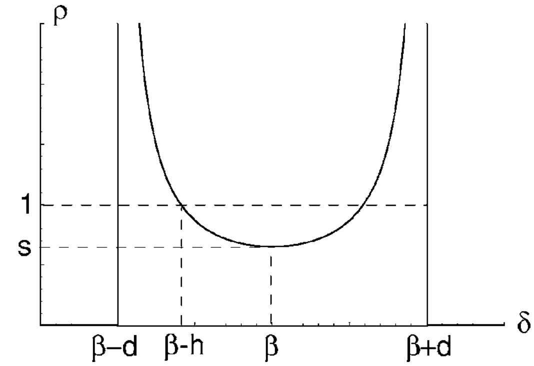

FIG. 1. Dependence of the eigenvalue ρ given by (35) associated with the eigenfunction $\mathcal{R}(m)$ (27), as a function of δ for the self-similar Vere-Jones model.

$$s = \frac{2\kappa}{d} \quad (36)$$

is going to play an important and physically intuitive role in the sequel. Note the major novelty compared with the standard ETAS model; here, we obtain a continuous spectrum of eigenvalues $\rho(\delta)$ rather that the unique one (28) associated with $\delta = a$ for the ETAS model.

In expression (35), since $1 - (\delta - \beta)^2 / d^2 \leqslant 1$, there is a solution with $\rho(\delta) \leqslant 1$ associated with the subcritical and critical regimes only for $s \leqslant 1$. For $s > 1$, all eigenvalues $\rho(\delta)$ are larger than 1, which corresponds to the explosive supercritical regime. We show below that the parameter s plays the role of an average branching ratio.

For $s \leqslant 1$, the continuous spectrum of eigenvalues is indexed by δ spanning the interval $[\beta - h, \beta]$, for which $\rho(\delta) \leqslant 1$ characterizes the subcritical and critical regimes. The exponent h is determined by the condition $\rho(\beta - h) = 1$, whose geometrical determination is represented in Fig. 1. We rule out the possibility $\delta > \beta$, which leads to unphysical solutions as shown below. Given this spectrum of eigenvalues, the growth of $\mathcal{R}(m)$ given by (27) is controlled by the largest eigenvalue $\rho(\beta - h) = 1$, when it exists, which leads to

$$\mathcal{R}(m) \sim e^{(\beta-h)m}, \quad h = d\sqrt{1-s}. \quad (37)$$

We can now describe the phase diagram of the self-similar Vere-Jones model, shown in Fig. 2.

(i) For $s > 1$, all eigenvalues $\rho(\delta)$ are larger than 1, corresponding to the explosive supercritical regime.

(ii) For $s < 1$ and $d < \beta$, we accept (37) as a physically appropriate eigenfunction only if it is monotonically increasing with respect to m, that is, if $0 \leqslant h \leqslant \beta$. This leads to the condition

$$\frac{1}{2}\left(d - \frac{\beta^2}{d}\right) < \kappa < \frac{d}{2}. \quad (38)$$

In the range (38) of parameters, there is always a solution, whatever the value $0 \leqslant s \leqslant 1$ of the form (37), associated with the unit eigenvalue. This is the critical regime, which is associated with a finite range of parameters $0 \leqslant s \leqslant 1$ and 0

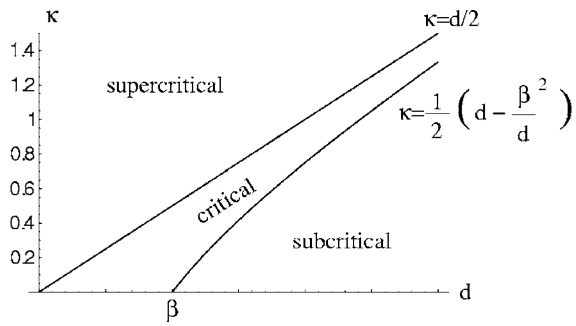

FIG. 2. Phase diagram in the plane of parameters (κ, d) at fixed β, for the self-similar Vere-Jones model, showing the domains of existence of the subcritical (39), critical (38), and supercritical ($s>1$) regimes.

$\leqslant h \leqslant \beta$ corresponding to the domain (38), indicated in Fig. 2.

(iii) For

$$0<\kappa<\frac{1}{2}\left(d-\frac{\beta^2}{d}\right), \tag{39}$$

which is only possible if $d>\beta$, we obtain $h>\beta$, that is, $\delta<0$, which corresponds to a seismic activity which is a decreasing function of the mother magnitude. This is the hallmark of the subcritical regime. We will see below that the average of the total number of offsprings is finite in this subcritical regime.

Figure 2 summarizes this phase diagram and delineates the domains of existence of the subcritical, critical, and supercritical regimes in the plane of parameters (κ, d) for a given β.

The physical meaning of this classification is obtained by examining the equation for the average number $\langle R\rangle(m,\mu)$ of aftershocks which are triggered by some mainshock with magnitude m. This equation is expression (15) written for the self-similar Vere-Jones model,

$$\langle R\rangle(m,\mu)=\int_{-\infty}^{\infty}\lambda(m-m')\langle R\rangle(m',\mu)dm'+\langle R_1\rangle(m,\mu), \tag{40}$$

where

$$\langle R_1\rangle(m,\mu)=\int_{\mu}^{\infty}\lambda(m-m')dm' \tag{41}$$

is the average of the number of corresponding first-generation aftershocks. Let us introduce the auxiliary function

$$S(m,\mu)=-\frac{\partial\langle R\rangle(m,\mu)}{\partial\mu}. \tag{42}$$

Using (40), $S(m,\mu)$ is solution of

$$S(m,\mu)=\int_{-\infty}^{\infty}\lambda(m-m')S(m',\mu)dm'+\lambda(m-\mu). \tag{43}$$

For its structure, it is clear that the solution of this equation depends only on the difference between m and μ, so that we write

$$S(m,\mu)=S(m-\mu). \tag{44}$$

The value of $\langle R\rangle(m,\mu)$ is obtained from $S(m,\mu)$ by using

$$\langle R\rangle(m,\mu)\equiv\int_{\mu}^{\infty}S(m,\mu')d\mu'=\int_{-\infty}^{m-\mu}S(x)dx. \tag{45}$$

Let us solve the equation

$$S(m)=\int_{-\infty}^{\infty}\lambda(m-m')S(m')dm'+\lambda(m) \tag{46}$$

for the auxiliary function $S(m)$. Applying the two-sided Laplace transform to this equation yields an equation for the Laplace transform

$$\hat{S}(u)=\int_{-\infty}^{\infty}S(m)e^{-um}dm \tag{47}$$

which takes the following form:

$$\hat{S}(u)=\frac{\hat{\lambda}(u)}{1-\hat{\lambda}(u)}. \tag{48}$$

The Laplace transform $\hat{\lambda}(u)$ of the kernel $\lambda(m)$ given by (6) can be explicitly calculated as

$$\hat{\lambda}(u)=\frac{2\kappa d}{d^2-(u-\beta)^2}. \tag{49}$$

Substituting expression (49) into (48) yields

$$\hat{S}(u)=\frac{2\kappa d}{h^2-(u-\beta)^2},\quad h^2=d^2-2\kappa d. \tag{50}$$

Its inverse Laplace transform

$$S(m)=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\hat{S}(u)e^{um}du \tag{51}$$

gives finally

$$S(m)=\frac{\kappa d}{h}e^{\beta m-h|m|}. \tag{52}$$

We can now use (52) in (45) to characterize the average of the number of aftershocks in the subcritical (39), critical (38) and supercritical ($s>1$) regimes.

(i) In the subcritical case $h>\beta$, we can take $\mu\to-\infty$ and still obtain a finite limit

$$\langle R\rangle=\lim_{\mu\to-\infty}\langle R\rangle(m,\mu)=\int_{-\infty}^{\infty}S(x)dx=\frac{2\kappa d}{h^2-\beta^2}<\infty \tag{53}$$

for the average of the total number of aftershocks of all generations triggered by an arbitrary spontaneous source. Thus, in the subcritical regime, the averages of total number of both observable and unobservable events are finite.

(ii) In the critical regime (38), we find that the average $\langle R\rangle(m,\mu)$ of the total number of events above any finite magnitude threshold μ is finite. In contrast, the average of the total number of aftershocks (including the unobservable tiny events of magnitudes $m'\to-\infty$) becomes infinite, $\langle R\rangle=\infty$. It is remarkable that we have at the same time $\langle R\rangle(m,\mu)$ finite for any $\mu>-\infty$ and $\langle R\rangle=\infty$. In real data, we only observe $\langle R\rangle(m,\mu)$. The underlying criticality is thus unobservable, due to the special cancellations in the self-similar Vere-Jones model, the infinite swarm of tiny events form an unobservable sea of activity, whose observable consequences lie in the finite activity at finite magnitudes. Thus, in a sense, this critical regime (38) can actually be decomposed into an effective subcritical regime for $s<1$ and a critical point reached at $s=1$ for observable events. Indeed, as $s\to 1$ (i.e., if $h\to 0$), expressions (45) and (52) for the average number of observable aftershocks triggered by an arbitrary spontaneous source tends to infinity. This shows that the critical regime for observable events corresponds to $s=1$ and confirms the interpretation of s as the effective branching ratio for observable events.

(iii) As $s\to 1$ $(h\to 0)$, $\langle R\rangle(m,\mu)$ tends to infinity for any m and μ, confirming that the supercritical regime corresponds to $s>1$.

D. Criticality condition for the generalized Vere-Jones ETAS model

We now analyze the critical conditions for the generalized Vere-Jones ETAS model defined by expression (2) in Sec. II A. It is convenient for the analysis to represent the intensity $\lambda(m,m')$ in (2) by

$$\lambda(m,m') = e^{a(m-m')}\nu(m,m'), \tag{54}$$

where

$$\nu(m,m') = \kappa e^{(a-\beta)m'-d|m-m'|}. \tag{55}$$

Let us introduce the auxiliary function

$$\mathcal{U}(m) = \mathcal{R}(m)e^{-am}, \tag{56}$$

and rewrite the homogeneous equation (24) in the form

$$\rho\mathcal{U}(m) = \int_{m_0}^{\infty}\mathcal{U}(m')\nu(m,m')dm'. \tag{57}$$

From the definition of $\nu(m,m')$ in (55), we see that the following equality holds:

$$\frac{d^2\nu(m,m')}{dm^2} = d^2\nu(m,m') - 2\kappa d\, e^{(a-\beta)m}\delta(m'-m), \tag{58}$$

where $\delta(x)$ is the Dirac-delta function. By differentiating equation (57) twice with respect to m, the identity (58) leads to the condition that, if $\mathcal{U}(m)$ is a solution of Eq. (57), then it should also satisfy the differential equation

$$\rho\frac{d^2\mathcal{U}(m)}{dm^2} = \left(\rho d^2 - 2\kappa d e^{(a-\beta)m}\right)\mathcal{U}(m), \quad m>m_0. \tag{59}$$

Thus, to determine the eigenvalue of Eq. (57), our strategy is to search for a solution of Eq. (59), which at the same time satisfies the integral condition

$$\rho\mathcal{U}(m_0) = \kappa\int_{m_0}^{\infty}\mathcal{U}(m)e^{(a-\beta)m-d(m-m_0)}dm, \tag{60}$$

which is derived from (57). It is straightforward to check that this strategy leads to solving (57). In addition, we need to impose that the eigenfunction $\mathcal{R}(m)$ is monotonically increasing as a function of m such that the condition (25) is satisfied. In terms of the auxiliary function $\mathcal{U}(m)$, this implies that

$$\lim_{m\to\infty}\mathcal{U}(m) < \infty. \tag{61}$$

Thus, our problem is to find a solution of (59) with the two conditions (60) and (61).

Before using this formalism for the generalized Vere-Jones ETAS model, it is instructive to see how it performs on the standard ETAS model (corresponding to $d=0$). For $d=0$, Eq. (59) reduces to

$$\rho\frac{d^2\mathcal{U}(m)}{dm^2} = 0. \tag{62}$$

Its solution satisfying condition (61) is $\mathcal{U}=C=$const. Substituting it into the integral relation (60) recovers the known expression for the unique eigenvalue equal to the average branching ratio,

$$\rho = \frac{\kappa e^{(a-\beta)m_0}}{\beta-a} = \frac{\gamma}{\gamma-1}\kappa', \quad \gamma=\frac{\beta}{a}. \tag{63}$$

The critical regime corresponds to the set of parameters obeying

$$\kappa' = \mathcal{K}(a,\beta) = 1-\frac{a}{\beta}, \tag{64}$$

while the domain of subcritical regime corresponds to $\kappa<\mathcal{K}(a,\beta)$. Similarly, to explore the condition for criticality in the generalized Vere-Jones ETAS model, we just need to put $\rho=1$ in (59) and search for the function

$$\kappa = \mathcal{K}(a,\beta,d) \tag{65}$$

such that the homogeneous equation

$$\frac{d^2\mathcal{U}(m)}{dm^2} = (d^2 - 2\kappa d e^{(a-\beta)m})\mathcal{U}(m), \quad m>m_0, \tag{66}$$

has a nontrivial solution increasing with m, which satisfies the integral equality (60) expressed for $\rho=1$. The subcritical regime then corresponds to the set of parameters such that $\kappa<\mathcal{K}(a,\beta,d)$.

In order to solve (66) for the critical case $\rho=1$, we introduce the new function $v(y)$ such that

$$\mathcal{U}(m) = v(y), \tag{67}$$

with

$$y = \varrho e^{-(m-m_0)d/\epsilon}, \quad \varrho = \frac{\sqrt{8\kappa' d\beta}}{\beta - a}, \quad \epsilon = \frac{2d}{\beta - a}. \tag{68}$$

Note that the parameter ϵ quantifies the transition from the ETAS model (obtained for $\epsilon=0$) to the self-similar Vere-Jones model (obtained for $\epsilon \to +\infty$). For small (respectively, large) ϵ, the generalized Vere-Jones ETAS model is close to the standard ETAS (respectively, self-similar Vere-Jones) model. Equation (66) for $\mathcal{U}(m)$ translates into the following Bessel equation for $v(y)$:

$$y^2 \frac{d^2 v(y)}{dy^2} + y \frac{dv(y)}{dy} + (y^2 - \epsilon^2) v(y) = 0. \tag{69}$$

It follows from (61) and from the definition (68) of y that the solution of (69) must satisfy.

$$\lim_{y \to 0} v(y) < \infty. \tag{70}$$

The integral condition (60) imposes in addition that

$$v(\varrho) = \frac{1}{2} \epsilon^{-1} \varrho^{-\epsilon} \int_0^{\varrho} v(y) y^{\epsilon+1} dy. \tag{71}$$

The solution of Eq. (69) has the form of a Bessel function

$$v(y) = A J_\epsilon(y), \tag{72}$$

where A is a constant. Substituting (72) into (71) and using the known recursion relation

$$\int y^{\epsilon+1} J_\epsilon(y) dy = y^{\epsilon+1} J_{\epsilon+1}(y) \tag{73}$$

between Bessel functions, we obtain the implicit equation for the variable ϱ,

$$J_\epsilon(\varrho) = \frac{\varrho}{2\epsilon} J_{\epsilon+1}(\varrho), \tag{74}$$

which determines the set of parameters corresponding to criticality ($\rho=1$).

Further insight can be obtained by determining the leading behavior of ϱ for small ϵ (quasi-ETAS model) and large ϵ (quasi-self-similar Vere-Jones model). For this, we introduce the new auxiliary parameter

$$\psi = \frac{\varrho}{\epsilon} = \sqrt{\frac{2\kappa' \beta}{d}}, \tag{75}$$

and rewrite Eq. (74) in the form

$$\psi = \frac{2 J_\epsilon(\epsilon\psi)}{J_{\epsilon+1}(\epsilon\psi)}. \tag{76}$$

For $\epsilon \ll 1$, we use the expansion

$$\frac{2J_\epsilon(x)}{J_{\epsilon+1}(x)} \simeq \frac{4}{x}(1+\epsilon) - \frac{x}{2+\epsilon} \quad (x \ll 1) \tag{77}$$

which gives the solution

$$\psi \simeq \sqrt{\frac{2}{\epsilon}(2+\epsilon)} \quad (\epsilon \ll 1). \tag{78}$$

In term of the original parameters κ, a, and d, this gives

$$\kappa' \simeq 1 - \frac{a}{\beta} + \frac{d}{\beta} \tag{79}$$

as the relation expression the critical regime $\rho=1$ of the generalized Vere-Jones ETAS model. Expression (79) differs from its counterpart (64) obtained for the standard ETAS model by the correction term d/β, which describes a kind of broadening of the subcritical regime due to the magnitude localization effect ($d>0$).

In the other limit $\epsilon \gg 1$ corresponding to the quasi-Vere-Jones model, the solution of the criticality condition (76) can also be obtained asymptotically by expanding the Bessel functions in the neighborhood of their first zero, i.e., by searching for $\epsilon\rho$ close to $v=v(\epsilon)$ defined by $J_\epsilon[v(\epsilon)]=0$. The corresponding expansions in Taylor series of the Bessel functions are

$$J_\epsilon(x) \simeq J'_\epsilon(v)(x-v), \quad J_{\epsilon+1}(x) \simeq J_{\epsilon+1}(v) + J'_{\epsilon+1}(v)(x-v). \tag{80}$$

It is well-known from the theory of Bessel functions that

$$J'_\epsilon(v) = -J_{\epsilon+1}(v), \quad J'_{\epsilon+1}(v) = -\frac{\epsilon+1}{v} J_{\epsilon+1}(v). \tag{81}$$

Thus, for $\epsilon \gg 1$, expressions (80) and (81) allow us to replace the criticality condition (76) by its asymptotic expression

$$\psi \simeq \frac{2v(v - \epsilon\psi)}{v + (\epsilon+1)(v - \epsilon\psi)}. \tag{82}$$

Solving this equation in ψ yields

$$\psi \simeq \frac{v(\epsilon)}{1+\epsilon} \quad (\epsilon \gg 1). \tag{83}$$

Using the known asymptotic relation

$$v(\epsilon) \simeq \epsilon + 1.856\ \epsilon^{1/3} + 1.033 \epsilon^{-1/3} \tag{84}$$

for the first zero of the Bessel function of large order $\epsilon \gg 1$, the criticality condition formulated in terms of the original parameters κ, a, d then reads

$$\kappa' \simeq \frac{d}{2\beta} + \frac{0.58}{\beta} d^{1/3} (\beta - a)^{2/3}. \tag{85}$$

The first leading term $\kappa' \approx d/2\beta$ recovers the critical condition $s=2\kappa/d=1$ for the self-similar Vere-Jones model. The last term on the rhs of (85) thus provides the first correction to the exactly self-similar Vere-Jones model when $a \neq \beta$. Figure 3 plots the numerical solution of Eq. (76) as a function of ϵ together with its two asymptotics regimes (78) and (83) with (84).

V. DISTRIBUTION OF EARTHQUAKE MAGNITUDES FOR VERE-JONES'S SELF-SIMILAR MODEL

The Gutenberg-Richter distribution of earthquake magnitudes is perhaps the most ubiquitous and documented statis-

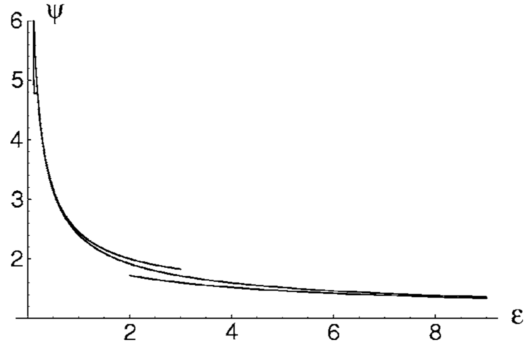

FIG. 3. Numerical solution of Eq. (76) as a function of ϵ defined in (68) compared with its two asymptotics regimes (78) and (83) with (84), giving the condition for criticality $\rho=1$ for the generalized Vere-Jones ETAS model.

tical property of earthquake catalogs. It is thus of great interest to investigate the prediction of Vere-Jones self-similar model for this quantity. We analyze in turn two versions of these statistics, first for the set of events triggered by a unique ancestor (a single earthquake of known magnitude) and second when summing over all possible main-shock magnitudes as in a large catalog. The first statistics is useful for understanding the cascade of triggered events but is not directly observed in real catalogs, as it is not possible to fully isolate an aftershock sequence. The second statistics refers to the standard measurements in catalogs including many aftershock sequences and can be directly compared with empirical data. The measurement of the Gutenberg-Richter distribution of earthquake magnitudes has been performed many times for what is believed to be a well-defined aftershock sequence associated with a single large mainshock. But due to the very long memory of the Omori law, it is extremely difficult to prove that so-called aftershocks are solely triggered by the mainshock and are not affected by the cumulative impact of all previous earthquakes. In addition, earthquakes trigger other earthquakes at large distances and delineating the spatial domain over which to construst the statistics of so-called aftershocks of a mainshock is an open problem. It is for all these reasons that the first statistics is not directly observable in a real catalog while the second one is. In order to study the statistical properties of a single aftershock sequence, sophisticated declustering methods are needed, which could be based on the idea of stochastic declustering [16,17] or on a network construction [18].

A. Distribution of the magnitude of aftershocks triggered by a mainshock of given magnitude m

Reasoning as in Sec. III B shows that the PDF's of the magnitudes μ of first-generation aftershocks and of all aftershocks of all generations of a mainshock of magnitude m are proportional, respectively, to $S_1(m,\mu)$ and $S(m,\mu)$ given by

$$S_1(m,\mu) = -\frac{\partial\langle R_1\rangle(m,\mu)}{\partial\mu}, \quad S(m,\mu) = -\frac{\partial\langle R\rangle(m,\mu)}{\partial\mu}. \tag{86}$$

Their expressions are

$$S_1(m,\mu) = \kappa e^{-\beta(\mu-m)-d|\mu-m|}, \tag{87}$$

$$S(m,\mu) = \frac{\kappa d}{h} e^{-\beta(\mu-m)-h|\mu-m|}. \tag{88}$$

Thus, the two branches $\mu<m$ with exponent $\beta-d$ and $\mu>m$ with exponent $\beta+d$ of the PDF $S_1(m,\mu)$ of first-generation aftershocks are renormalized in two other branches of the same form, with d renormalized into $h=d\sqrt{1-s}$. Even if one could isolate a single aftershock sequence (and we stressed above the associated difficulties), only $S(m,\mu)$ would be observed since one cannot easily distinguish between the different generations of triggered aftershocks. As mentioned above, specific declustering algorithms could be used to reconstruct $S(m,\mu)$ from real catalogs [16–18]. If observations confirm that a single exponential distribution is a good description of the distribution of aftershock magnitudes over all generations triggered from a single mainshock, then expression (88) shows that the PDF $S(m,\mu)$ may be quite close to a single exponential law, even if the difference between the two branches of $S_1(m,\mu)$ is significant, as long as the critical parameter s is close to 1 [for $s\to 1, S(m,\mu)\sim e^{-\beta\mu}$ and is a pure exponential law]. In other words, an observable PDF of aftershock magnitudes close to a single exponential law would be compatible with strong deviations from a pure exponential law for first-generation events, due to the renormalization effect over all the generations which effectively mixes up the two branches sufficiently close to criticality $s\to 1$. If this was the case, this would join previous studies of the standard ETAS model [10,11] and suggest that the earth is operating close to a critical point. But this remains to be tested in a future work.

B. Distribution of earthquake magnitudes over all events

The Gutenberg-Richter law for the distribution of the magnitudes of earthquakes is generally a statistical property established for a large space-time-magnitude domain, without restrictions. It is thus interesting to ask what Vere-Jones' self-similar model predicts for the distribution of magnitude of a large stationary sequence of events triggered by a steady-state influx of spontaneous sources. The answer is given by expression (22) for $f(\mu,m_d)$. We thus need to make explicit the dependence of $f(\mu,m_d)$ on the parameters of the model. This problem depends on four key parameters, χ is the exponent for spontaneous sources defined in (19); β is a crucial parameter in the definition of the PDF of magnitudes of first-generation aftershocks; h and s appear in the condition for criticality.

Let us first investigate the contribution $g(\mu)$ of the events triggered by the spontaneous source. For Vere-Jones' self-similar model, we have

$$g(\mu) = \int_{m_s}^{\infty} S(m-\mu)p(m)dm. \tag{89}$$

Substituting (19) and (52) in (89) yields

$$g(\mu) = p(\mu)K(m_s - \mu, \beta - \chi), \tag{90}$$

where $K(m_s-\mu,\beta-\chi)$ describes the deviation of the PDF of the magnitudes of triggered events from the PDF of the magnitudes of the spontaneous sources. This function is given by

$$K(y,z) = \frac{\kappa d}{h}\int_y^{\infty} e^{zx-h|x|}dx. \tag{91}$$

Its explicit expression is

$$K(y,z) = \frac{2\kappa d}{h^2 - z^2} - \frac{\kappa d}{h(h+z)}e^{(z+h)y}. \tag{92}$$

The corresponding complementary cumulative distribution, i.e., the total number of triggered events with magnitude larger than μ, is equal to

$$\langle R\rangle(\mu) = \int_{\mu}^{\infty} g(m)dm = \int_{\mu}^{\infty} p(m)K(m_s - m, \beta - \chi)dm. \tag{93}$$

We need to distinguish three cases as $g(\mu)$ given by (90) is qualitatively different for different values χ,β, and h.

(1) For $\chi<\beta-h$, $g(\mu)=\infty$ for any μ. This results from the fact that the spontaneous sources with large magnitudes dominate the production of triggered events in this case. This supercritical regime can be tamed with the introduction of an upper magnitude cutoff m_{max} but is not investigated further here. The limiting case $\chi=\beta-h$ gives a criticality condition for the observable events in the framework of the self-similar Vere-Jones model.

(2) Consider the regime

$$\beta - h < \chi < \beta + h, \tag{94}$$

and the limit $m_s\to-\infty$ corresponding to $y\to-\infty$, for which $K(y,z)$ has the following asymptotic dependence:

$$K(y,z) \simeq \frac{2\kappa d}{h^2 - z^2}, \quad y \to -\infty. \tag{95}$$

The corresponding asymptotic expressions for the PDF $g(\mu)$ given by (90) and the complementary cumulative distribution given by (93) are

$$g(\mu) \simeq \frac{2\kappa d}{h^2 - z^2}p(\mu), \quad \langle R\rangle(\mu) \simeq \frac{2\kappa d}{h^2 - z^2}Q(\mu). \tag{96}$$

Substituting (96), (19), and (20) into (22) yields the distribution of the magnitudes of all observable events (in the limit $m_s\to-\infty$)

$$f(\mu,m_d) = \chi e^{-\chi(m-m_d)} \quad (\mu > m_d). \tag{97}$$

Remarkably, in this regime (94), the observed Gutenberg-Richter distribution is predicted to reveal only the exponent of the distribution of the spontaneous sources and to be blind to the exponents β,d of the distribution of triggered events. In other words, the PDF of the magnitudes of all observed events reproduces that of the spontaneous sources given by (19).

Substituting (96) into (23) yields the fraction of triggered events whose magnitudes are larger than μ, among all analogous events,

$$n(\mu) = n = \frac{2\kappa d}{d^2 - (\beta - \chi)^2}, \tag{98}$$

which is found independent of μ. If the same distribution of magnitudes describes the spontaneous sources and the triggered events ($\beta=\chi$), then $n=s$. Thus, the critical regime corresponds to $s=1$, confirming the interpretation of s as equivalent to the critical branching ratio of Vere-Jones' self-similar model. However, if $\beta\neq\chi$ and $|\beta-\chi|\to h$, then we obtain $n\to1$ even for $s<1$ ($h>0$), essentially all the events are triggered.

(3) Consider the regime

$$\chi > \beta + h. \tag{99}$$

In this case, the function K is closely approximated by its asymptotic behavior

$$K(y,z) \simeq -\frac{\kappa d}{h(h+z)}e^{(z+h)y}. \tag{100}$$

The corresponding asymptotic behaviors of $g(\mu)$ given by (90) and $\langle R\rangle(\mu)$ given by (93) for large $\mu-m_s$ are

$$g(\mu) \simeq \frac{\kappa d\chi}{h(\chi - \beta - h)}e^{-(\beta+h)(\mu-m_s)} \tag{101}$$

and

$$\langle R\rangle(\mu) \simeq \frac{\kappa d\chi}{h(\chi - \beta - h)(\beta + h)}e^{-(\beta+h)(\mu-m_s)}. \tag{102}$$

Substituting these two expression into (22) and (23) yields

$$f(\mu,m_d) \simeq (\beta + h)e^{-(\beta+h)(\mu-m_d)}, \quad n \simeq 1, \tag{103}$$

asymptotically for $\mu-m_s\to+\infty$.

VI. SUMMARY AND DISCUSSION

Our results can be summarized as follows.

(1) We have clarified and quantified the conditions under which the self-similar Vere-Jones model as well as a more general version (which contains both the standard ETAS and Vere-Jones version as special cases) are critical, subcritical, and supercritical. Only the subcritical and critical regimes give a stationary process in the presence of a nonzero flux of immigrants.

(2) We have shown that the concept of an average branching ratio s, defined as the average number of daughters of first generation per mother of magnitudes above a finite magnitude threshold) holds for Vere-Jones model in a broad domain of parameters. Remarkably, s is found independent of the magnitude threshold used. However, the average branching ratio loses its meaning when the magnitude threshold is pushed to $-\infty$ (in other words, when it is removed), as the existence of arbitrary small events allowed in this model dominates and makes the average divergent. Since empirical catalogs are always characterized by a minimum magnitude m_d of completeness, our results apply directly and show that

it is possible to have an infinite number of—unobservable but still important for the cascade of triggering—events per mother together with a finite average branching ratio for observable events.

(3) Vere-Jones' model is defined by the Gutenberg-Richter (GR) magnitude distribution for first-generation events triggered by a source of magnitude m having two branches: for aftershocks magnitudes $m' < m$, the GR exponent is $\beta - d$, while it is $\beta + d$ for $m' > m$. We have shown that, accounting for the contributions of all generations of triggered events, this GR distribution is renormalized into another two-branches law, for aftershocks magnitudes $m' < m$, the renormalized GR exponent is $\beta - h$, while it is $\beta + h$ for $m' > m$, where $0 \leq h = d\sqrt{1-s} \leq d$ for $0 \leq s \leq 1$.

(4) By suitable declustering techniques, it is in principle possible to obtain relatively robust determinations of sequences of aftershocks associated with a single mainshock. In such sequences, only the renormalized GR would be observable. If it was confirmed that the distributions of aftershocks are single exponentials, assuming that Vere-Jones model is a correct description, this would imply that h is small so that the difference between the exponents $\beta - h$ and $\beta + h$ is within the empirical uncertainties and variations from sequences to sequences. This would imply that either d is small or s is close to 1 (condition describing the boundary between the subcritical to supercritical regimes of the Vere-Jones model) or both. We note again that the prediction that two distinct exponents h and d characterize, respectively, the GR law over all generations and the GR law over the first generation of triggered events could be in principle tested empirically using an adaptation of the statistical declustering method developed by Zhuang *et al.* [16,17] to Vere-Jones's model. In this respect, Zhuang *et al.* [17] have already found that the magnitude distribution of the triggered event depends on the magnitude of its direct ancestor, with an exponent smaller for large events (in contradiction with other less sophisticated studies using more arbitrary space-time windows [19,20]): this is roughly consistent with our prediction with Vere-Jones model, as the observable distribution for large (respectively, small) ancestors is weighted more by the $m' < m$ (respectively, $m' > m$) regime associated with exponent $\beta - h$ (respectively, $\beta + h$). But a systematic statistical study is needed to ascertain this conclusion.

(5) The distribution of magnitudes over a stationary catalog (obtained by summing over an average steady flow of spontaneous sources) is found universal (independent of β, d and the other parameters) and a pure GR with exponent equal to the exponent χ of the spontaneous sources (which in full generality is allowed to be different from the exponent β involved in the distribution of triggered events) in a large domain of the parameter space. This implies that, if the exponent of the spontaneous sources is different from the exponent of triggered events, the physics of cascades of triggering in the self-similar Vere-Jones model implies that only the former exponent is observable in global catalogs. Again, the statistical declustering method of Zhuang *et al.* [16,17] should be able to test this prediction. In this respect, we note that Zhuang *et al.* find that the background events have a larger exponent than the triggered events, $\chi > \beta$.

Finally, as mentioned in the introduction, the main limitation of this (otherwise analytically exact) study has been to integrate over space and time. In the higher dimensional space-time version, it cannot be excluded that some delicate conditions for stability (subcritical and critical regimes) established here might need to be revised to account for new phenomena appearing in higher dimensions. This will be investigated in a future work.

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